Stochastic Models of Higher-Order Networks

Point Processes and Topological Data Analysis

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2. Random Connection Hypergraphs
3. Functional Convergence of Edge Counts in Dynamic RCMs

(Topology of Higher-Order Random Connection Models)

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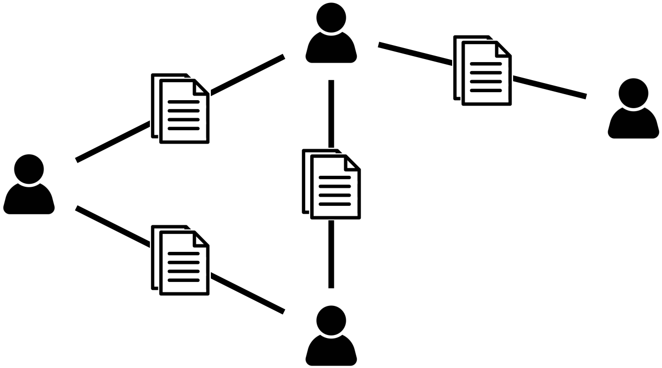
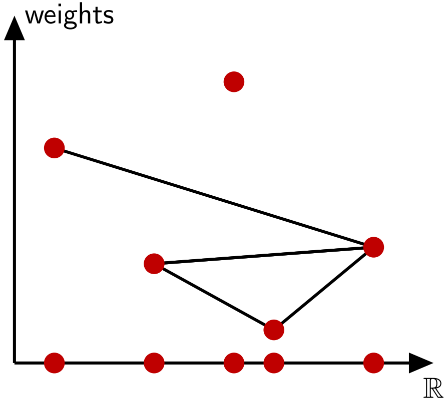
Complex Systems



Weighted Random Connection Models

- **Common properties**
  - power-law distribution
  - small-world property
  - community structure (clustering)
- **Model**
  - vertices: point process on a metric space
  - weights: i.i.d. random variables
  - edges: random connection function



M. Deijfen, R. van der Hofstad, and G. Hooghiemstra. Scale-free percolation. *Ann. Inst. Henri Poincaré Probab. Stat.*, 49(3):817–838, 2013.

Kernel-Based Random Connection Model

- vertices: point process $\mathcal{P}_s$ on $\mathbb{R}^d$ with intensity $s > 0$
- weights: i.i.d. random variables W_x, distribution $F_W(w)$
- interpolation kernel $\kappa\colon [0,\infty) \times [0,\infty) \to [0,\infty)$

$$\kappa(w_1, w_2) := (w_1 \wedge w_2)(w_1 \vee w_2)^a \qquad (a \geqslant 0)$$

- connection probability: $\varphi\colon [0,\infty) \to [0,1]$

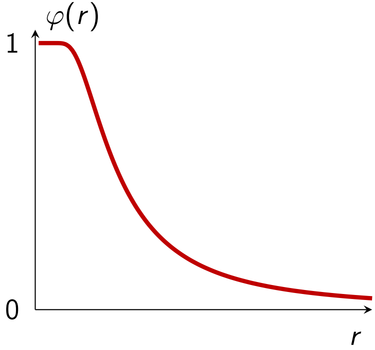
$$\varphi\Big(\frac{|B_{|x-y|}(o)|}{v_s\,\kappa(W_x, W_y)}\Big) \qquad \int_0^\infty \varphi(r)\,\mathrm{d}r = 1$$

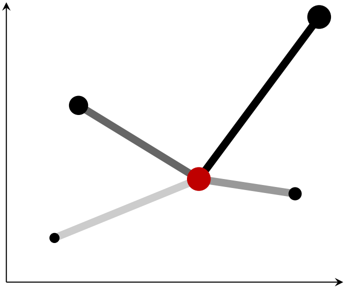
- regularity conditions on φ and F_W:

$$\lim_{r\uparrow\infty} \frac{\varphi(tr)}{\varphi(r)} = t^{-\alpha} \qquad \lim_{w\uparrow\infty} \frac{1 - F_W(tw)}{1 - F_W(w)} = t^{-\beta} \quad \alpha > 1,\ \beta \geqslant a\alpha$$

Example

$$\varphi(r) := 1 - \exp(-r^{-2}/\pi) \qquad \kappa(w_1, w_2) := w_1 w_2$$





Goal of the Study

- Spatial distribution of k-degree vertices:

$$\xi_s := \sum_{x \in \mathcal{P}_s \cap [0,1]^d} \mathbb{1}\{\deg(x) = k\} \, \delta_x$$

- Goal: approximate ξ_s as $s \to \infty$
- Specify scaling v_s such that $\mathbb{E}[\xi_s([0,1]^d)] = 1$
- Typical weight of a degree-k node:

$$\mathbb{P}(W \leqslant w_s) = 1/(2s)$$

Poisson Approximation

Theorem (Hirsch, Jahnel, Jhawar, J.; 2025)

ζ: *Poisson point process with intensity* $\mathbb{1}\{x \in [0,1]^d\}$

If

$$\begin{aligned} (sv_s)^{-1} w_s^{-\eta} &\in o(1/\log(s)) \\ F(w_s^{\eta})\, w_s^{-(K+1)(1-\eta)} &\in o(1/\log(s)) \\ w_s^{-(1-K\delta)(1-\eta)} &\in o(1/\log(s)). \end{aligned}$$

Then

$$d_{\mathrm{KR}}(\xi_s, \zeta) = \sup_{h \in \mathrm{LIP}(1)} \left\{ \left| \mathbb{E}[h(\xi_s)] - \mathbb{E}[h(\zeta)] \right| \right\} \xrightarrow[s\uparrow\infty]{} 0$$

M. D. Penrose. Inhomogeneous random graphs, isolated vertices, and Poisson approximation. *J. Appl. Probab.*, 55(1):112–136, 2018.

S. K. Iyer and S. K. Jhawar. Poisson approximation and connectivity in a scale-free random connection model. *Electron. J. Probab.*, 26(23):Paper No. 86, 2021.

Stein's Method

O. Bobrowski, M. Schulte, and D. Yogeshwaran. Poisson process approximation under stabilization and Palm coupling. *Ann. H. Lebesgue*, 5:1489–1534, 2022.

- $(\mathbb{X}, \mathcal{X}), (\mathbb{Y}, \mathcal{Y})$: lcscH spaces
- $\boldsymbol{N}_{\mathbb{X}}$: set of σ-finite counting measures on $\mathbb{X}$
- $\mathcal{F}(\mathbb{X})$: set of closed subsets of $\mathbb{X}$
- $\mathcal{S} \colon \mathbb{X}^m \times \boldsymbol{N}_{\mathbb{X}} \to \mathcal{F}$: stopping sets
  $\{\omega \in \boldsymbol{N}_{\mathbb{X}} \colon \mathcal{S}(\boldsymbol{x}, \omega) \subseteq A\} = \{\omega \in \boldsymbol{N}_{\mathbb{X}} \colon \mathcal{S}(\boldsymbol{x}, \omega \cap A) \subseteq A\} \quad (\forall A \subseteq \mathbb{X})$
- Indicator
  $g \colon \mathbb{X}^m \times \boldsymbol{N}_{\mathbb{X}} \to \{0, 1\} \qquad g(\boldsymbol{x}, \omega) = g(\boldsymbol{x}, \omega \cap A) \quad \forall A \supset \mathcal{S}(\boldsymbol{x}, \omega)$
- Function
  $f \colon \mathbb{X}^m \times \boldsymbol{N}_{\mathbb{X}} \to \mathbb{Y} \qquad f(\boldsymbol{x}, \omega) = f(\boldsymbol{x}, \omega \cap A) \quad \forall A \supset \mathcal{S}(\boldsymbol{x}, \omega)$

Stein's Method

Theorem (Bobrowski, Schulte, Yogeshwaran; 2022)

- *η, ζ: Poisson point processes on $\mathbb{X}$ with intensities $\lambda_\eta, \lambda_\zeta < \infty$*
- *ξ: point process on $\mathbb{X}$ with intensity λ_ξ*

$$\xi[\eta] := \frac{1}{m!} \sum_{\boldsymbol{x} \in \eta^m_{\neq}} g(\boldsymbol{x}, \eta)\, \delta_{f(\boldsymbol{x},\eta)}$$

- $\hat{\mathcal{S}} : \mathbb{X}^m \to \mathcal{F} \qquad \boldsymbol{x} \subseteq \hat{\mathcal{S}}(\boldsymbol{x})$
- $g_{\mathrm{tr}}(\boldsymbol{x}, \omega) := g(\boldsymbol{x}, \omega)\, \mathbb{1}\{\mathcal{S}(\boldsymbol{x}, \omega) \subseteq \hat{\mathcal{S}}(\boldsymbol{x})\}$

Then

$$d_{\mathrm{KR}}(\xi, \zeta) \leqslant d_{\mathrm{TV}}(\lambda_\xi, \lambda_\zeta) + E_1 + E_2 + E_3 + E_4$$

Stein's Method

Theorem (Bobrowski, Schulte, Yogeshwaran; 2022)

$$E_1 := \frac{2}{m!} \int_{\mathbb{X}^m} \mathbb{E}\big[g(\boldsymbol{x}, \eta \cup \boldsymbol{x})\, \mathbb{1}\{\mathcal{S}(\boldsymbol{x}, \eta \cup \boldsymbol{x}) \not\subset \hat{\mathcal{S}}(\boldsymbol{x})\}\big]\, \lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{x})$$

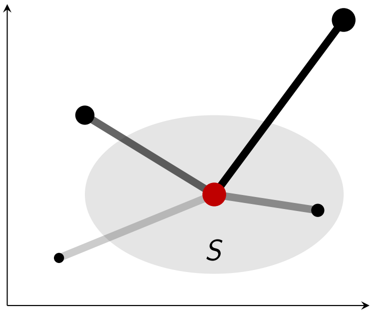
$$E_2 := \frac{2}{(m!)^2} \iint_{\mathbb{X}^m \times \mathbb{X}^m} \mathbb{1}\{\hat{\mathcal{S}}(\boldsymbol{x}) \cap \hat{\mathcal{S}}(\boldsymbol{z}) \neq \varnothing\} \times \mathbb{E}\big[g_{\mathrm{tr}}(\boldsymbol{x}, \eta \cup \boldsymbol{x})\big]\, \mathbb{E}\big[g_{\mathrm{tr}}(\boldsymbol{z}, \eta \cup \boldsymbol{z})\big]\, \lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{x}) \lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{z})$$

$$E_3 := \frac{2}{(m!)^2} \iint_{\mathbb{X}^m \times \mathbb{X}^m} \mathbb{1}\{\hat{\mathcal{S}}(\boldsymbol{x}) \cap \hat{\mathcal{S}}(\boldsymbol{z}) \neq \varnothing\} \times \mathbb{E}\big[g_{\mathrm{tr}}(\boldsymbol{x}, \eta \cup \boldsymbol{x} \cup \boldsymbol{z}) g_{\mathrm{tr}}(\boldsymbol{z}, \eta \cup \boldsymbol{x} \cup \boldsymbol{z})\big]\, \lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{x}) \lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{z})$$

$$E_4 := \frac{2}{m!} \sum_{\varnothing \subsetneq I \subsetneq \{1,\dots,m\}} \frac{1}{(m-|I|)!} \int_{\mathbb{X}^m} \int_{\mathbb{X}^{m-|I|}} \mathbb{E}\big[g_{\mathrm{tr}}(\boldsymbol{x}, \eta \cup \boldsymbol{x} \cup \boldsymbol{z}) \times g_{\mathrm{tr}}((\boldsymbol{x}_I, \boldsymbol{z}), \eta \cup \boldsymbol{x} \cup \boldsymbol{z})\big]\, \lambda_\xi^{\otimes m-|I|}(\mathrm{d}\boldsymbol{z}) \lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{x})$$

Proof Sketch

- Problem: g should be localized to $S \in \mathcal{F}$
- Step 1: mark approximation neglect high weighted points
- Step 2: reach approximation: neglect far points



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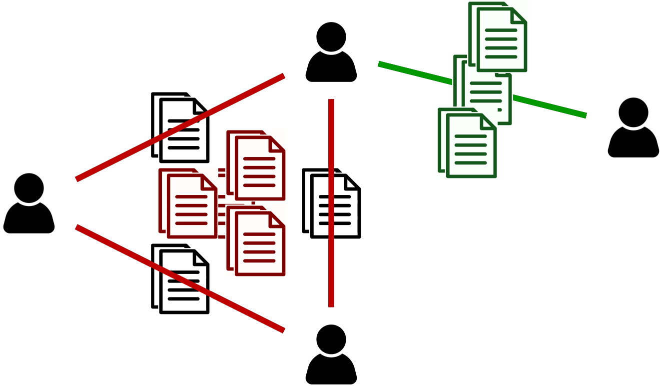
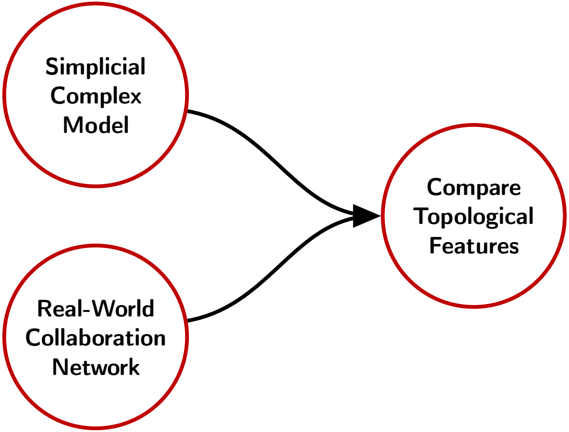
Complex Systems



Simple graphs: information loss

Outline


Simplicial Complex Model
Real-World Collaboration Network
Compare Topological Features

Random Connection Hypergraph Model (RCHM)

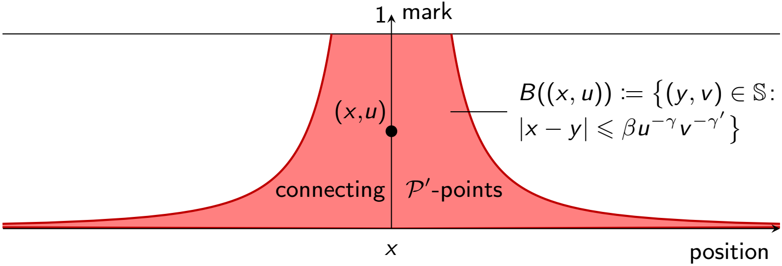
Two Poisson point processes on $\mathbb{S} := \mathbb{R} \times [0,1]$

- vertices: $\mathcal{P} \subset \mathbb{S}$
- hyperedges: $\mathcal{P}' \subset \mathbb{S}$

Connections:

$$\{(X,U),(Y,V)\} \in \mathcal{P} \times \mathcal{P}'$$

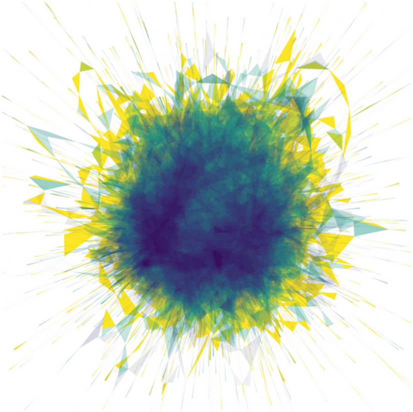
$$|X - Y| \leqslant \beta\, U^{-\gamma}\, V^{-\gamma'} \quad \gamma, \gamma' \in (0,1)$$



R. van der Hofstad, J. Komjáthy, and V. Vadon. Random intersection graphs with communities. *Adv. in Appl. Probab.*, 53(4):1061–1089, 2021.

Collaboration Network

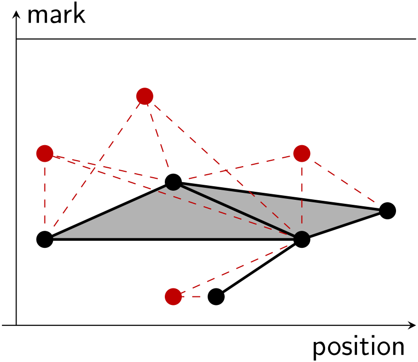
- Source: arXiv preprint server
- Vertices: $\sim 8 \cdot 10^5$ authors
- Hyperedges: $\sim 10^6$ documents

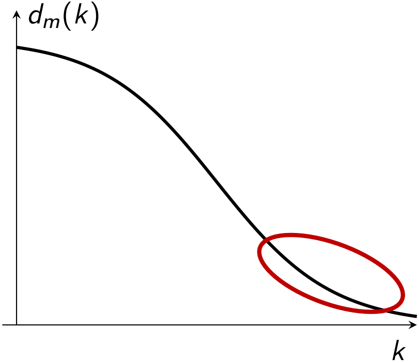


Statistics dataset (1990–2023)

Higher-Order Degree Distribution

$$\deg(\Delta) := \mathcal{P}'\Big(\bigcap_{p\in\Delta} B(p)\Big) \qquad d_m(k) := \mathbb{P}\big(\deg(\Delta_m^*) \geqslant k\big)$$





Higher-Order Degree Distributions

Theorem (Brun, Hirsch, J., Otto; 2025+)

$$\lim_{k\uparrow\infty} \log(d_m(k))/\log(k) = m - (m+1)/\gamma,$$

where

$$d_m(k) = \mathbb{P}\big(\deg(\Delta_m^*) \geqslant k\big).$$

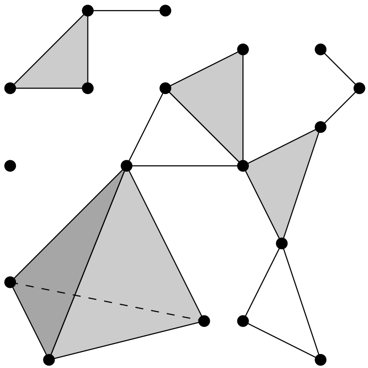
- Parameter fitting: model parameters can be tuned to obtain a given power-law exponent.

Simplicial Complex

Idea: represent higher-order relationships with a simplicial complex

Definition (Simplicial Complex)

- (P, Σ)
- $P \neq \varnothing$, $|P| < \infty$
- $\Sigma \subseteq 2^P$
- $p \in P \implies \{p\} \in \Sigma$
- $\sigma \in \Sigma$, $\Delta \subseteq \sigma \implies \Delta \in \Sigma$



RCHM = simplicial complex

$$\Sigma = \left\{\Delta \subseteq \mathcal{P} \colon \deg(\Delta) \geqslant 1\right\}$$

Betti Numbers

Definition (k-chains)

$$C_k(K) := \left\{ \sum_i a_i \sigma_i, : a_i \in \mathbb{Z}_2,\ \sigma_i \in \Sigma, |\sigma_i| = k+1 \right\}$$

Definition (Boundary operator)

$$\partial_k : C_k(K) \to C_{k-1}(K) \qquad\qquad \partial_k(\sigma_i) = \sum_{j=0}^{k} \sigma_i^{(j)}$$

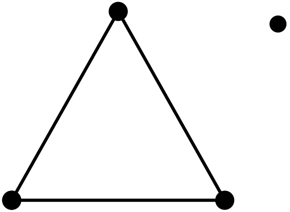
Definition (Cycle and boundary groups)

Cycle group: $Z_k = \mathrm{ker}(\partial_k)$

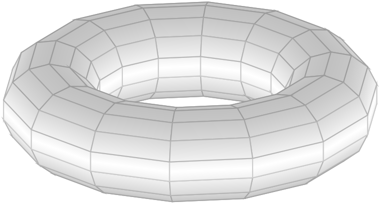
Boundary group: $B_k = \mathrm{im}(\partial_{k+1})$

kth Betti number: $\beta_k = \dim(Z_k) - \dim(B_k)$

Betti Numbers



$\beta_0 = 2;\ \beta_1 = 1;\ \beta_2 = 0$



$\beta_0 = 1;\ \beta_1 = 2;\ \beta_2 = 1$

CLT for Betti Numbers

- Window: $\mathbb{S}_n := [0, n] \times [0, 1]$
- Simplicial complex: $G_n := G(\mathcal{P} \cap \mathbb{S}_n, \mathcal{P}' \cap \mathbb{S}_n)$
- Betti number of dimension m in window n: $\beta_m^{(n)} := \beta_m(G_n)$

Theorem (Brun, Hirsch, J., Otto; 2025+)

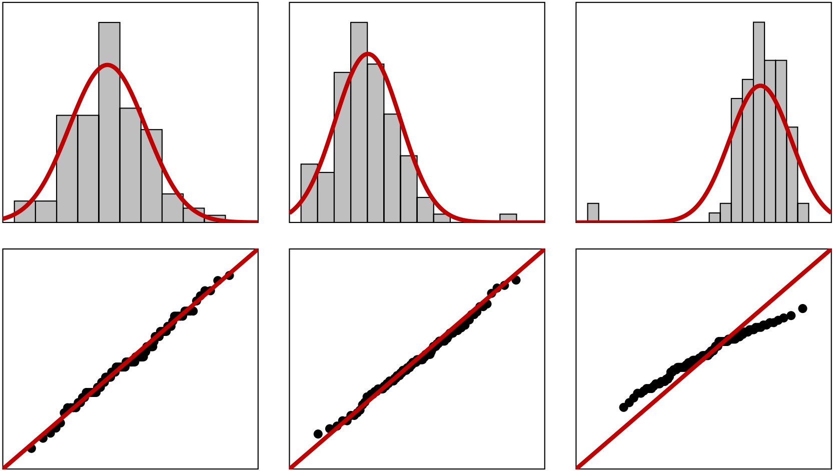
If $\gamma < 1/4$, $\gamma' < 1/(4(m+1))$, *then*

$$n^{-1/2}\big(\beta_m^{(n)} - \mathbb{E}[\beta_m^{(n)}]\big) \xrightarrow[n\uparrow\infty]{d} \mathcal{N}(0, \sigma^2) \qquad \sigma^2 \geqslant 0.$$

M. D. Penrose and J. E. Yukich. Central limit theorems for some graphs in computational geometry. *Ann. Appl. Probab.*, 11(4):1005–1041, 2001.

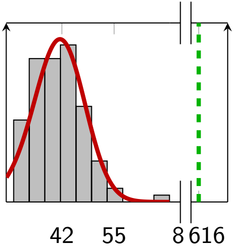
D. Pabst. Betti numbers in the random connection model for higher-dimensional simplicial complexes and the Boolean model. *arXiv preprint arXiv:2506.13429*, 2025.

Simulated Distribution of Betti-1 Numbers

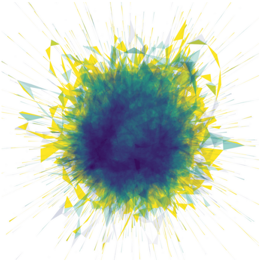


$\gamma = 0.25$ $\gamma = 0.50$ $\gamma = 0.75$

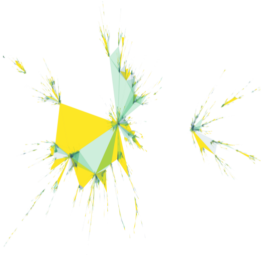
Hypothesis Test



Hypothesis test



Network data



Simulation

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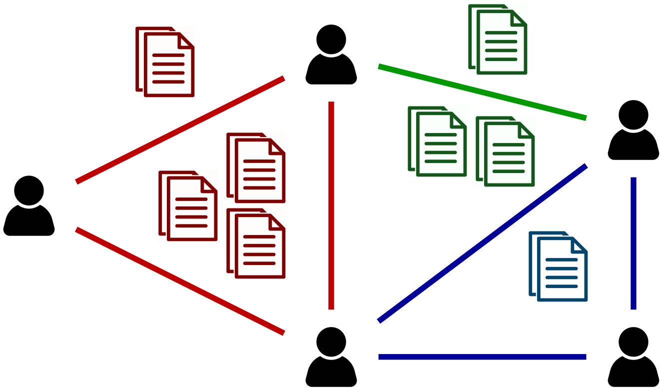
Complex Systems



Network dynamics?

Dynamic Random Connection Hypergraph Model

Vertices: $\mathcal{P}$		
position	$X \in \mathbb{R}$	Leb
mark	$U \in (0,1]$	Leb
birth time	$B \in \mathbb{R}$	Leb
lifetime	$L \in [0,\infty)$	Exp(1)

Hyperedges: $\mathcal{P}'$		
position	$Z \in \mathbb{R}$	Leb
mark	$W \in (0,1]$	Leb
time	$R \in \mathbb{R}$	Leb

Connection conditions:

spatial condition

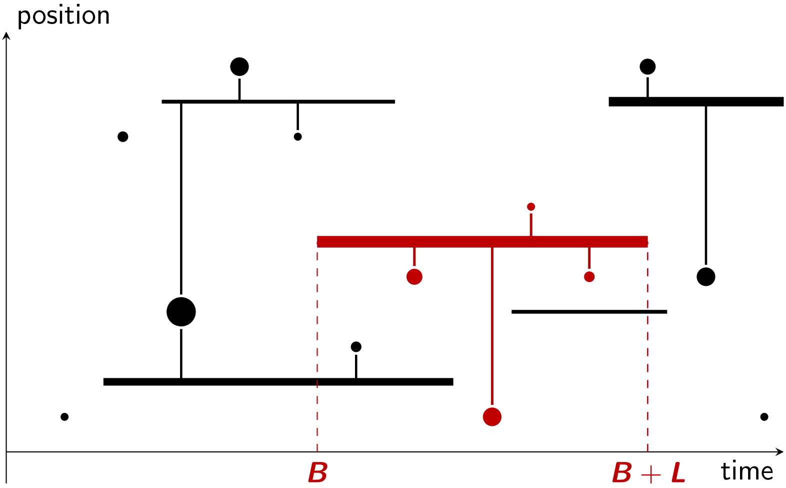
$$|X - Z| \leqslant \beta U^{-\gamma} W^{-\gamma'}$$

temporal condition

$$B \leqslant R \leqslant B + L$$

E. Onaran, O. Bobrowski, and R. J. Adler. Functional central limit theorems for local statistics of spatial birth–death processes in the thermodynamic regime. *Ann. Appl. Probab.*, 33(5):3958–3986, 2023.

Dynamic Random Connection Hypergraph Model



Edge counts

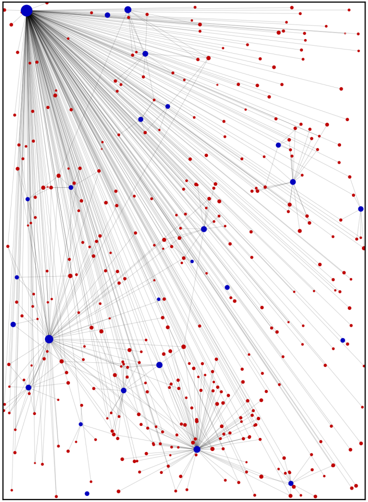
- Degree of $P := (X, U, B, L) \in \mathcal{P}$:

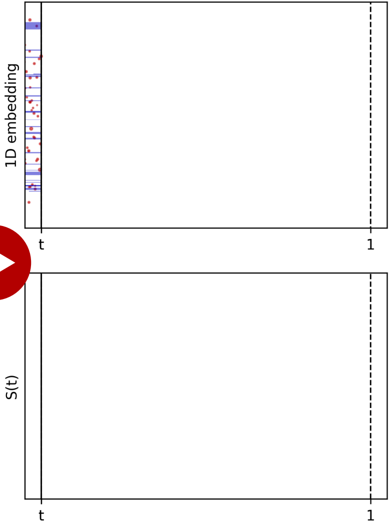
$$\deg(P; t) := \sum_{(Z,W,R)\in\mathcal{P}'} \mathbb{1}\left\{|X - Z| \leqslant \beta U^{-\gamma} W^{-\gamma'}\right\} \times \mathbb{1}\left\{B \leqslant R \leqslant t \leqslant B + L\right\}$$

- Edge count

$$S_n(\,\cdot\,) := \sum_{P\in\mathcal{P}} \deg(P; \,\cdot\,)\, \mathbb{1}\{X \in [0, n]\}$$

Edge counts





Finite Variance Case

Proposition (Hirsch, Jahnel, J.; 2025+)

If $\gamma < 1/2$ *and* $\gamma' \in (0,1)$*, then*

$$\overline{S}_n(t) := n^{-1/2}\big(S_n(t) - \mathbb{E}[S_n(t)]\big) \xrightarrow[n\uparrow\infty]{d} \mathcal{N}(0,\sigma^2) \quad \sigma^2 > 0.$$

Theorem (Hirsch, Jahnel, J.; 2025+)

G*: Gaussian process*

$$\mathrm{Cov}(G(t_1), G(t_2)) = (c_1 + c_2|t_1 - t_2|)\exp(-\mu|t_1 - t_2|)$$

If $\gamma, \gamma' < 1/4$*, then*

$$\overline{S}_n(\,\cdot\,) \xrightarrow[n\uparrow\infty]{d} G(\,\cdot\,).$$

Univariate Stable Limit Theorem

Theorem (Hirsch, Jahnel, J.; 2025+)

If $\gamma > 1/2$, *then*

$$\overline{S}_n(t) := n^{-\gamma}\big(S_n(t) - \mathbb{E}[S_n(t)]\big) \xrightarrow[n\uparrow\infty]{d} \mathcal{S}(1/\gamma).$$

Functional Convergence of Edge Counts

- $\nu([\varepsilon,\infty)) := c\varepsilon^{-1/\gamma}$: Lévy measure on $[0,\infty)$
- $\mathcal{P}_\infty := \mathrm{PPP}(\nu \otimes \mathrm{Leb}_{\mathbb{R}} \otimes \mathrm{Exp}(1))$: Poisson point process on $[0,\infty) \times \mathbb{R} \times \mathbb{R}_+$

$$S_\varepsilon^*(\cdot) := \sum_{(J,B,L)\in\mathcal{P}_\infty} J(\cdot - B)\,\mathbb{1}\{J \geqslant \tilde{c}\varepsilon^\gamma\}\,\mathbb{1}\{B \leqslant \cdot \leqslant B + L\}$$

$$\overline{S}_\infty(\cdot) := \lim_{\varepsilon\downarrow 0} \overline{S}_\varepsilon^*(\cdot) := \lim_{\varepsilon\downarrow 0}\big(S_\varepsilon^*(\cdot) - \mathbb{E}[S_\varepsilon^*(\cdot)]\big)$$

Theorem (Hirsch, Jahnel, J.; 2025+)

If $\gamma > 1/2$ and $\gamma' < 1/4$, then $\overline{S}_\infty(\cdot) \in D([0,1],\mathbb{R})$, and

$$\overline{S}_n(\cdot) := n^{-\gamma}\big(S_n(t) - \mathbb{E}[S_n(t)]\big) \xrightarrow[n\uparrow\infty]{d} \overline{S}_\infty(\cdot) \quad \textit{in } D([0,1],\mathbb{R}).$$

Proof of Functional SLT

Part 1

$$S_n(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \deg(P;\,\cdot\,)$$

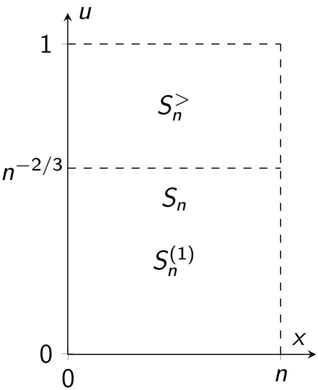
$$S_n^{>}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \deg(P;\,\cdot\,)\,\mathbb{1}\{U > n^{-2/3}\}$$

Proposition (Hirsch, Jahnel, J.; 2025+)

If $\gamma > 1/2$, $\gamma' < 1/4$, *then*

$$\overline{S}_n^{>}(\,\cdot\,) \xrightarrow[n\uparrow\infty]{d} 0 \qquad \text{in } D([0,1],\mathbb{R}).$$

$$S_n^{(1)}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \deg(P;\,\cdot\,)\,\mathbb{1}\{U \leqslant n^{-2/3}\}$$

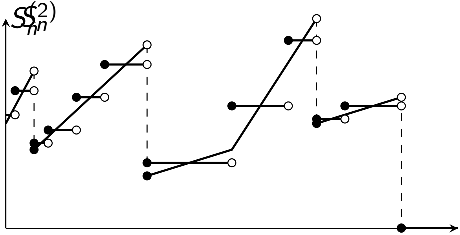


Part 1

$$S_n^{(2)}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \mathbb{E}\big[\deg(P;\,\cdot\,)\,\big|\,P\big]\,\mathbb{1}\{U \leqslant n^{-2/3}\}$$

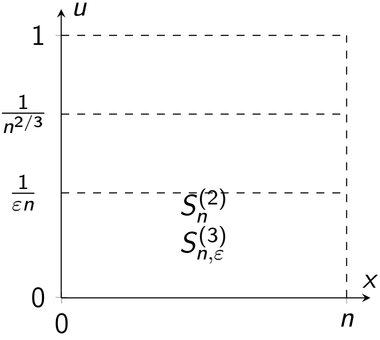
Proposition (Hirsch, Jahnel, J.; 2025+)

If $\gamma > 1/2,$ *then* $\left\| \overline{S}_n^{(1)} - \overline{S}_n^{(2)} \right\| \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0.$



Part 2 – Steps

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\overline{S}^{(3)}_{n,\varepsilon} \arrow[r, "n \to \infty", "\text{Step 4}"'] \arrow[d, "\varepsilon \to 0\ (\forall n)"', "\text{Step 3}"] & \overline{S}^{*}_{\varepsilon} \arrow[d, "\text{Step 5}"', "\varepsilon \to 0"] \\
\overline{S}^{(2)}_{n} \arrow[r, dashed, "\text{goal}"] & \overline{S}_{\infty}
\end{tikzcd}
$$



$$S^{(3)}_{n,\varepsilon}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \mathbb{E}[\deg(P;\,\cdot\,) \mid P]\, \mathbb{1}\{U \leqslant 1/(\varepsilon n)\}$$

Step 3

Goal:

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\uparrow\infty}\mathbb{P}\big(d_{\mathsf{Sk}}\big(\overline{S}^{(3)}_{n,\varepsilon},\overline{S}^{(2)}_{n}\big)>\delta\big)=0\qquad\text{for all }\delta>0$$

Proposition (Hirsch, Jahnel, J.; 2025+)

Let $\gamma > 1/2$ and $\gamma' \in (0,1)$. Then,

$$\limsup_{n\uparrow\infty}\big\|\overline{S}^{(3)}_{n,\varepsilon}-\overline{S}^{(2)}_{n}\big\|\xrightarrow[\varepsilon\downarrow 0]{\mathbb{P}}0.$$

Step 4

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\overline{S}^{(3)}_{n,\varepsilon} \arrow[r, "n \to \infty", "\text{Step 4}"'] \arrow[d, color=green, "\varepsilon \to 0 \quad (\forall n)\text{ unif.}"', "\text{Step 3}"] & \overline{S}^{*}_{\varepsilon} \arrow[d, "\text{Step 5}"', "\varepsilon \to 0"] \\
\overline{S}^{(2)}_{n} \arrow[r, dashed, "\text{goal}"] & \overline{S}_{\infty}
\end{tikzcd}
$$

Step 4

Notation: $N_s(p) := \{(z, w) \in \mathbb{R} \times (0,1] : |x - z| \leqslant \beta u^{-\gamma} w^{-\gamma'}\}$

In $S_n^{(3)}$: $\quad U \leqslant 1/(\varepsilon n) \quad \Longleftrightarrow \quad n^{-\gamma}|N_s(P_s)| \geqslant \tilde{c}\varepsilon^{\gamma}$

Lemma (Hirsch, Jahnel, J.; 2025+)

Let $\gamma > 1/2$ *and* $\gamma' \in (0,1)$. *Then,*

$$n\,\mathbb{P}\big(n^{-\gamma}|N_s(P)| \in \cdot\,\big) \xrightarrow[n\uparrow\infty]{v} \nu(\,\cdot\,) \qquad \nu([a,\infty)) := \tilde{c}^{1/\gamma} a^{-1/\gamma}.$$

S. I. Resnick. *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling.* Springer, New York, 2007.

Step 4 – Convergence of Point Processes

$$\sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \delta_{(n^{-\gamma}|N_{\mathsf{s}}(P_{\mathsf{s}})|, B, L)} \xrightarrow[n\uparrow\infty]{v} \mathcal{P}_\infty,$$

where

$$\mathcal{P}_\infty := \mathrm{PPP}(\nu \otimes \mathrm{Leb}_{\mathbb{R}} \otimes \mathrm{Exp}(1)) \quad \text{on } \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$$

Step 4 – Summation Functional

- χ summation functional: point process $\mapsto$ edge count process:

$$\chi(\eta)(\,\cdot\,) := \sum_{(j,b,\ell)\in\eta} j(t-b)\,\mathbb{1}\{j \geqslant \tilde{c}\varepsilon^{\gamma}\}\,\mathbb{1}\{b \leqslant \,\cdot\, \leqslant b+\ell\}$$

- Limiting edge count process $S^*_\varepsilon := \chi(\mathcal{P}_\infty)$

Proposition (Hirsch, Jahnel, J.; 2025+)

The summation functional χ is almost surely continuous with respect to the distribution of $\mathcal{P}_\infty$.

Lemma (Hirsch, Jahnel, J.; 2025+)

Let $\gamma > 1/2$. Then $\mathbb{E}[S^*_\varepsilon] < \infty$ and $\overline{S}^{(3)}_{n,\varepsilon}(\,\cdot\,) \xrightarrow[n\uparrow\infty]{d} \overline{S}^*_\varepsilon(\,\cdot\,)$.

Step 5

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overline{S}^{(3)}_{n,\varepsilon} \arrow[r, "n \to \infty", "\text{Step 4}"'] \arrow[d, "\varepsilon \to 0 \quad (\forall n)\ \text{unif.}"', "\text{Step 3}"] & \overline{S}^{*}_{\varepsilon} \arrow[d, "\text{Step 5}"', "\varepsilon \to 0"] \\
\overline{S}^{(2)}_{n} \arrow[r, dashed, "\text{goal}"] & \overline{S}_{\infty}
\end{tikzcd}$$

Step 5 – Almost Sure Convergence of $\overline{S}^*_\varepsilon$

Proposition (Hirsch, Jahnel, J.; 2025+)

Let $\varepsilon_k \to 0$ *be a decreasing sequence. Then,*

$$\sup_{n,m \geqslant N} \left\| \overline{S}^*_{\varepsilon_n} - \overline{S}^*_{\varepsilon_m} \right\| \xrightarrow[N\uparrow\infty]{\text{a.s.}} 0.$$

Step 5

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\overline{S}^{(3)}_{n,\varepsilon} \arrow[r, "n \to \infty", "\text{Step 4}"'] \arrow[d, "\varepsilon \to 0 \quad (\forall n) \text{ unif.}"', "\text{Step 3}"] & \overline{S}^{*}_{\varepsilon} \arrow[d, "\text{Step 5}"', "\varepsilon \to 0"] \\
\overline{S}^{(2)}_{n} \arrow[r, dashed, "\text{goal}"] & \overline{S}_{\infty}
\end{tikzcd}
$$

Summary

- **Complex Systems**
  - Crucial multibody relationships
  - Higher-order networks, simplicial complexes
- **Goal**
  - Stochastic models for higher-order networks
- **Research Projects**
  - Poisson Approximation in WRCMs
  - (Topology of Higher-Order ADRCM)
  - Random Connection Hypergraph Model
  - Functional Convergence in Dynamic RCHM

Thank you for your attention!

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