Scaling limits in spatial interacting particle systems

Péter Juhász

Department of Mathematics
Aarhus University

Danish Data Science Academy
Copenhagen, Denmark

June 17, 2025

Related Publication

Christian Hirsch

Benedikt Jahnel

Péter Juhász

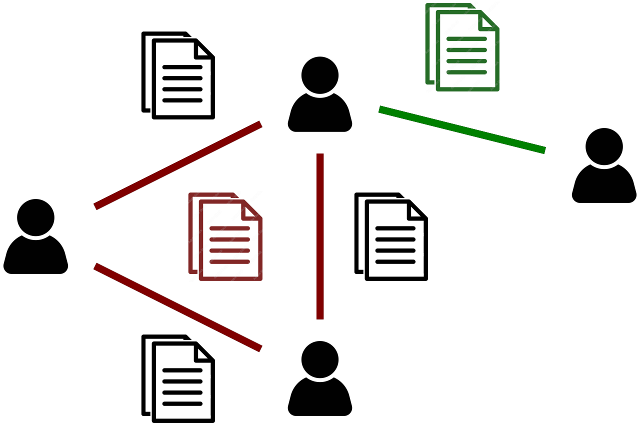
C. Hirsch, B. Jahnel, and P. Juhász

Functional limit theorems for edge counts
in dynamic random connection hypergraphs

Table of Contents

1. Motivation
2. Dynamic Hypergraph Model
3. Edge Counts
4. Functional Convergence of Edge Counts

Complex Systems



Simple graphs: information loss

Model

Goal

- develop a network model
- study its scaling limits

Model

- bipartite graph: vertices, hyperedges
- representation: two Poisson point processes $\mathcal{P}$ and $\mathcal{P}'$

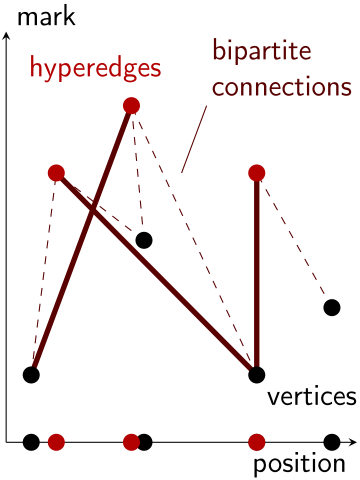
Vertices: $\mathcal{P}$

- position: $X \in \mathbb{R}$
- mark: $U \in [0, 1]$
- birth time: $B \in \mathbb{R}$
- lifetime: $L \in [0, \infty)$

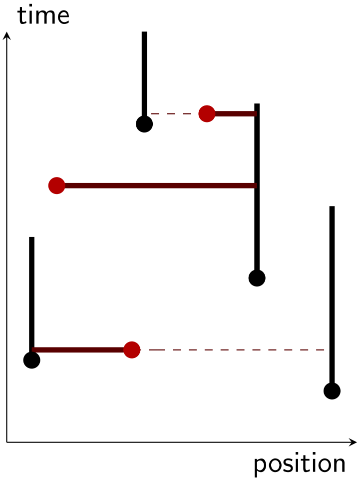
Hyperedges: $\mathcal{P}'$

- position: $Y \in \mathbb{R}$
- mark: $V \in [0, 1]$
- interaction time: $I \in \mathbb{R}$

Dynamic Random Connection Hypergraph Model



(a) Spatial condition

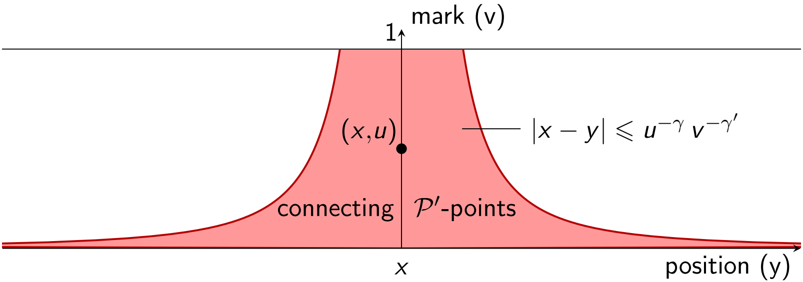


(b) Temporal condition

Spatial Connection Condition

$$P = (X, U, B, L) \in \mathcal{P}$$
$$P' = (Y, V, I) \in \mathcal{P}'$$

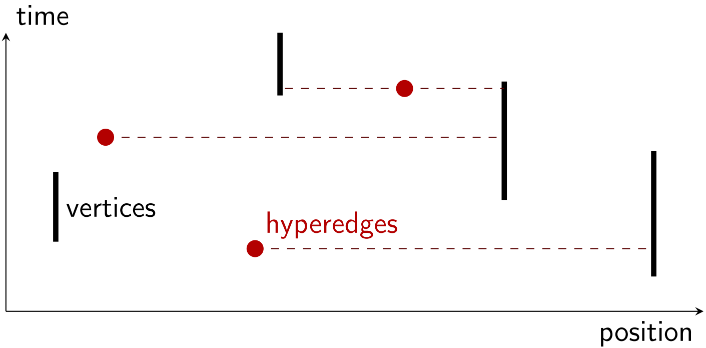
spatial condition: $|X - Y| \leqslant U^{-\gamma} V^{-\gamma'}$ $\gamma, \gamma' \in (0, 1)$



Temporal Connection Condition

$$P = (X, U, B, L) \in \mathcal{P}$$
$$P' = (Y, V, I) \in \mathcal{P}'$$

temporal condition: $B \leqslant I \leqslant B + L$



Edge counts

- State space:

$$\left.\begin{array}{l}\mathbb{S} := \mathbb{R} \times [0,1] \\ \mathbb{T} := \mathbb{R} \times \mathbb{R}_+\end{array}\right\} \implies \left\{\begin{array}{l}\mathcal{P} \subseteq \mathbb{S} \times \mathbb{T} \\ \mathcal{P}' \subseteq \mathbb{S} \times \mathbb{R}\end{array}\right.$$

- Degree of $P := (X, U, B, L) \in \mathcal{P}$:

$$\begin{aligned}\deg(P; t) := \sum_{(Y,V,I)\in\mathcal{P}'} & \mathbb{1}\left\{|X - Y| \leqslant U^{-\gamma} V^{-\gamma'}\right\} \\ & \times \mathbb{1}\left\{B \leqslant I \leqslant t \leqslant B + L\right\}\end{aligned}$$

- Edge count

$$S_n(\,\cdot\,) := \sum_{P\in\mathcal{P}} \deg(P;\,\cdot\,)\,\mathbb{1}\{X \in [0, n]\}$$

Finite Variance Domain

- If $\gamma < 1/2$, then $\mathrm{Var}(S_n(t)) < \infty$ for all $t \in [0,1]$, and the univariate CLT holds:

$$\overline{S}_n(t) := n^{-1/2}(S_n(t) - \mathbb{E}[S_n(t)]) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

- G: Gaussian process

$$\mathrm{Cov}(G(t_1), G(t_2)) = (c_1 + c_2|t_1 - t_2|)\exp(-\mu|t_1 - t_2|)$$

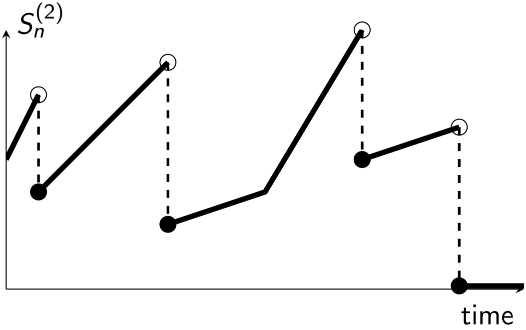
Theorem (Hirsch, Jahnel, J., 2025)

If $\gamma, \gamma' < 1/4$, then the edge count process $\overline{S}_n(\,\cdot\,)$ converges weakly to $G(\,\cdot\,)$ as $n \to \infty$.

Functional Convergence of Edge Counts

If $\gamma > 1/2$, then $\mathrm{Var}(S_n(t)) = \infty$ for all $t \in [0,1]$, and the univariate SLT holds:

$$\overline{S}_n(t) := n^{-\gamma}(S_n(t) - \mathbb{E}[S_n(t)]) \xrightarrow{d} \mathcal{S}(1/\gamma)$$



Functional Stable Limit Theorem

- $\nu([\varepsilon, \infty)) := c\varepsilon^{-1/\gamma}$: Lévy measure on $\mathbb{J} := [0, \infty)$
- $\mathcal{P}_\infty := \mathrm{PRM}(\mathbb{J} \times \mathbb{T})$: with intensity measure $\nu \times \mathrm{Leb}_{\mathbb{R}} \times \mathrm{Exp}(1)$

$$\overline{S}_\infty(\cdot) := \lim_{\varepsilon \downarrow 0} \Big(\sum_{(J,B,L) \in \mathcal{P}_\infty} J\, \mathbb{1}\{J \geqslant \varepsilon\}(\cdot - B)\, \mathbb{1}\{\cdot \in [B, B+L]\} - c'\, \varepsilon^{-(1/\gamma - 1)} \Big)$$

Theorem (Hirsch, Jahnel, J., 2025)

If $\gamma > 1/2$ and $\gamma' < 1/4$, then $\overline{S}_\infty(\cdot) \in D([0,1], \mathbb{R})$ exists, and the edge count process $\overline{S}_n(\cdot)$ converges weakly to $\overline{S}_\infty(\cdot)$ in the Skorokhod space $D([0,1], \mathbb{R})$ as $n \to \infty$.

Thank you for your attention!

This work was supported by the Danish Data Science Academy, which is funded by the Novo Nordisk Foundation (NNF21SA0069429) and Villum Fonden (40516).

References

- Y. Davydov, *Weak convergence of discontinuous processes to continuous ones*, Probability theory and mathematical statistics (1996), 15–18.
- J. D. Esary, F. Proschan, and D. W. Walkup, *Association of random variables, with applications*, Ann. Math. Statist. **38** (1967), no. 5, 1466 – 1474.
- P. Gracar, A. Grauer, L. Lüchtrath, and P. Mörters, *The age-dependent random connection model*, Queueing Systems **93** (2019), no. 3, 309–331.
- G. Last and M. D. Penrose, *Lectures on the Poisson process*, vol. 7, Cambridge University Press, 2017.
- S. I Resnick, *Heavy-tail phenomena: Probabilistic and statistical modeling*, Springer Science & Business Media, 2007.
- W. Whitt, *Stochastic-process limits*, Springer New York, NY, 2002.