Functional Stable Limit in Random Connection Hypergraphs

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1. Random Connection Hypergraph Model

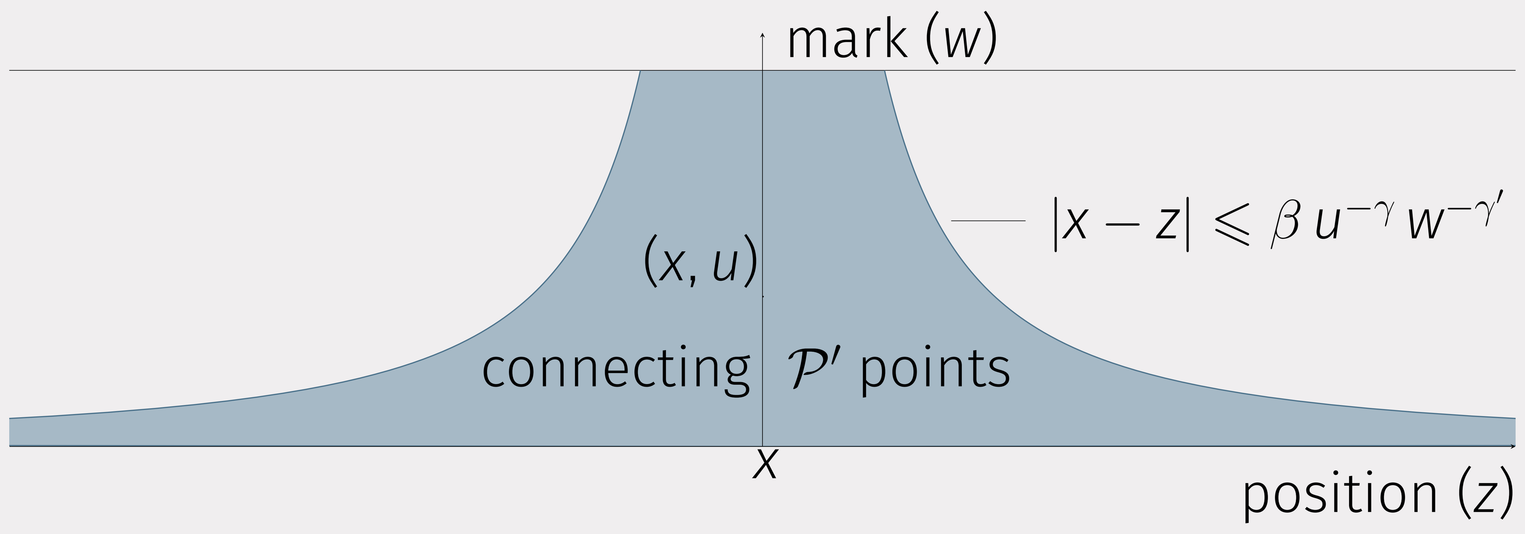
- **Hypergraph model**: random connection bipartite graph
- **Vertices**: Two marked, independent Poisson point processes:

$$\mathcal{P} = \{(X_i, U_i, B_i, L_i)\}_{i \geqslant 1} \quad \text{on} \quad \mathbb{S} \times \mathbb{T} := (\mathbb{R} \times [0,1]) \times (\mathbb{R} \times \mathbb{R}_+)$$
$$\mathcal{P}' = \{(Z_i, W_i, R_i)\}_{i \geqslant 1} \quad \text{on} \quad \mathbb{S} \times \mathbb{R} := (\mathbb{R} \times [0,1]) \times \mathbb{R}.$$

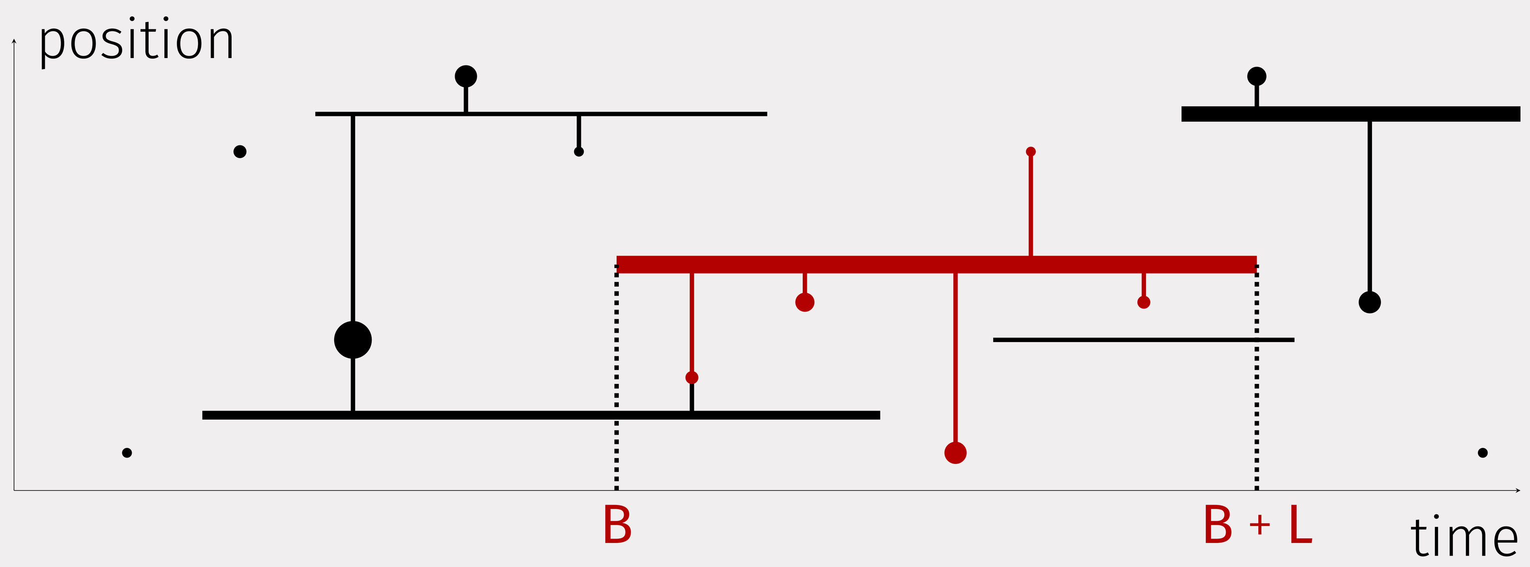
$\mathcal{P}$-points (nodes)			$\mathcal{P}'$-points (interactions)		
position	$X \in \mathbb{R}$	Leb	position	$Z \in \mathbb{R}$	Leb
weight	$U \in [0,1]$	Leb	weight	$W \in [0,1]$	Leb
birth time	$B \in \mathbb{R}$	Leb	interaction time	$R \in \mathbb{R}$	Leb
lifetime	$L \in [0,\infty)$	Exp(1)			

- **Connection conditions:**

spatial condition: $|X - Z| \leqslant \beta U^{-\gamma} W^{-\gamma'}$ $\quad \beta > 0, \ \gamma, \gamma' \in (0,1)$
temporal condition: $B \leqslant R \leqslant B + L,$



Spatial connection condition [1]



Temporal connection condition

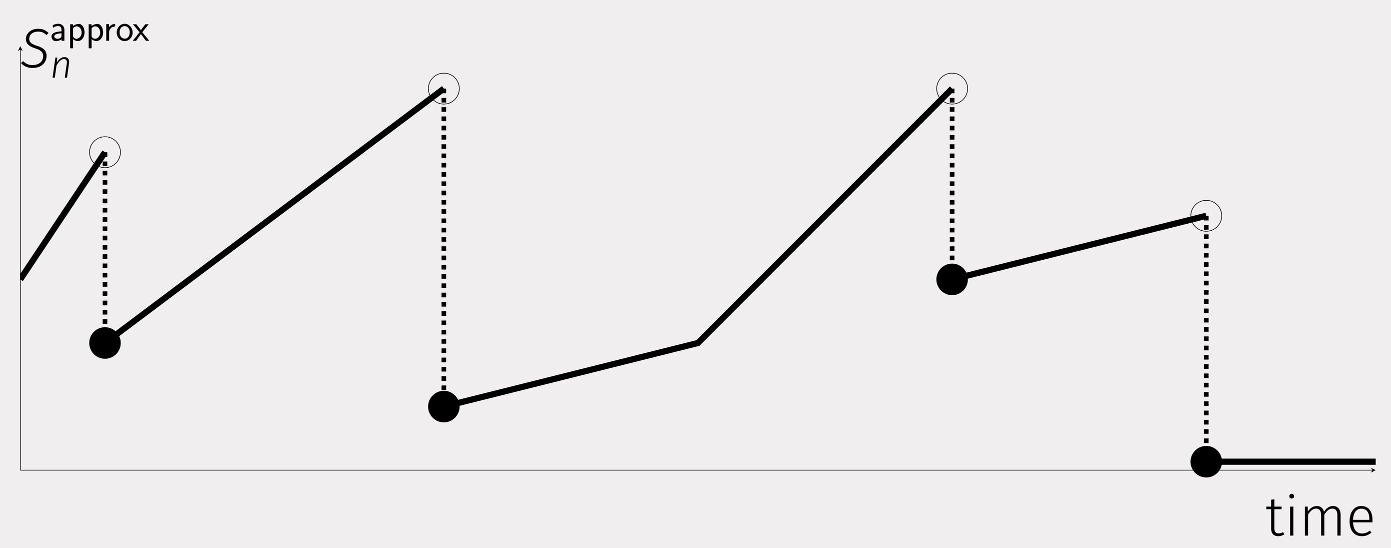
2. Edge Count Process

- Degree of $P \in \mathcal{P}$ at time t:

$$\deg(P; t) := \sum_{(Z,W,R) \in \mathcal{P}'} \mathbb{1}\{|X - Z| \leqslant \beta U^{-\gamma} W^{-\gamma'}\} \mathbb{1}\{B \leqslant R \leqslant t \leqslant B + L\}.$$

- Edge count process:

$$S_n(\,\cdot\,) := \sum_{P \in \mathcal{P}} \deg(P; \,\cdot\,) \mathbb{1}\{X \in [0, n]\}.$$



Approximation of the edge count process

3. Functional Stable Limit Theorem

Theorem (Functional stable limit of edge count).

- ν*: measure on* $\mathbb{J} := [0,\infty)$ *with* $\nu([\varepsilon, \infty)) := c\,\varepsilon^{-1/\gamma}$
- $\mathcal{P}_\infty$*: Poisson point process on* $\mathbb{J} \times \mathbb{T}$ *with intensity* $\nu \times \mathrm{Leb} \times \mathrm{Exp}(1)$

$$S_\varepsilon^*(\,\cdot\,) := \sum_{(J,B,L) \in \mathcal{P}_\infty} J(\,\cdot\, - B) \mathbb{1}\{J \geqslant \tilde{c}\varepsilon^{\gamma}\} \mathbb{1}\{B \leqslant \,\cdot\, \leqslant B + L\} - c(\varepsilon)$$

If $\gamma > 1/2$ *and* $\gamma' < 1/4$*, then, in the Skorokhod space* $D([0,1], \mathbb{R})$*,*

$$n^{-\gamma}(S_n(\,\cdot\,) - \mathbb{E}[S_n(\,\cdot\,)]) \xrightarrow[n \uparrow \infty]{d} \lim_{\varepsilon \downarrow 0} S_\varepsilon^*(\,\cdot\,).$$

Proof sketch.

Notations: normalized processes : $\overline{\,\cdot\,} := n^{-\gamma}(\,\cdot\, - \mathbb{E}[\,\cdot\,])$
observation window : $\mathbb{S}_n := [0, n] \times [0, 1]$

Step 1: High-mark edge count is negligible.

high-mark edge count: $S_n^{\gtrless}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap (\mathbb{S}_n \times \mathbb{T})} \deg(P; \,\cdot\,) \mathbb{1}\{U > n^{-2/3}\}$

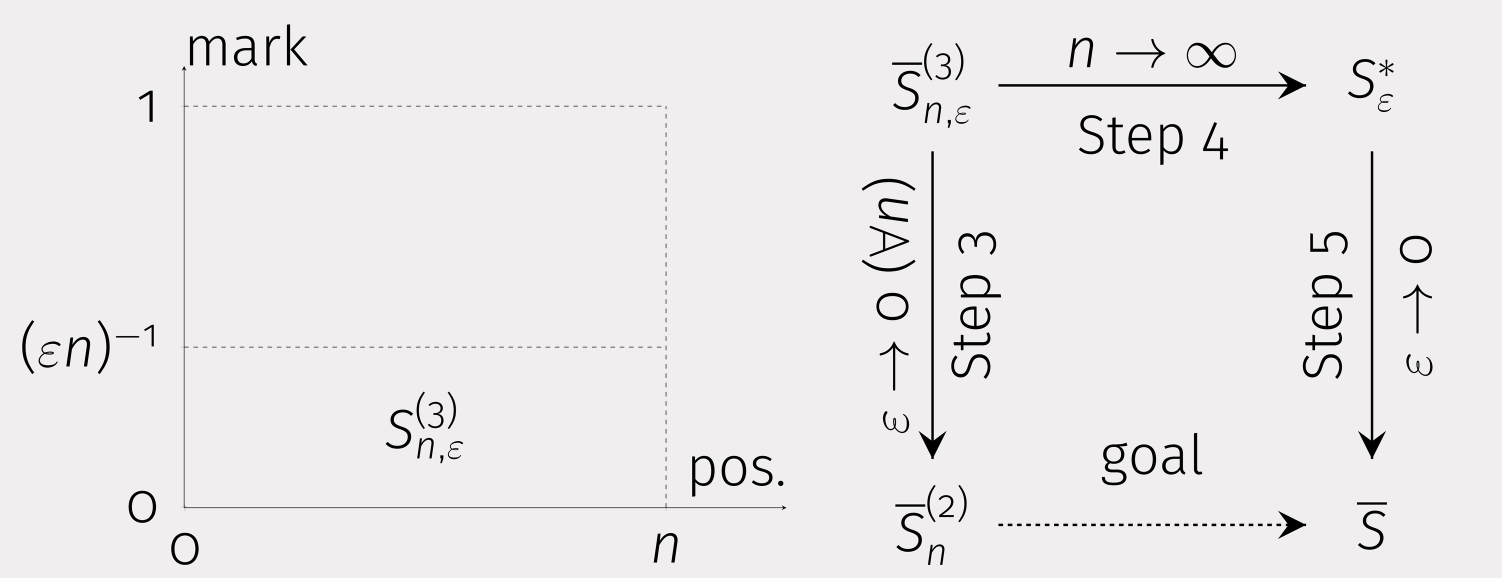
low-mark edge count: $S_n^{(1)}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap (\mathbb{S}_n \times \mathbb{T})} \deg(P; \,\cdot\,) \mathbb{1}\{U \leqslant n^{-2/3}\}$

$$n^{-\gamma}(S_n^{\gtrless}(\,\cdot\,) - \mathbb{E}[S_n^{\gtrless}(\,\cdot\,)]) \xrightarrow[n \uparrow \infty]{d} 0 \quad \text{in} \quad D([0,1], \mathbb{R}).$$

Step 2: Approximation of the low-mark edge count.

$$S_n^{(2)}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap (\mathbb{S}_n \times \mathbb{T})} \mathbb{E}\big[\deg(P; \,\cdot\,) \big| P\big] \mathbb{1}\{U \leqslant n^{-2/3}\} \qquad \|\overline{S}_n^{(1)} - \overline{S}_n^{(2)}\| \xrightarrow[n \uparrow \infty]{\mathbb{P}} 0$$

Result: $\overline{S}_n^{(2)}$ is devoid of spatial correlations of neighborhoods.



Step 3: Only the lowest-mark vertices contribute in the limit. [2]

$$S_{n,\varepsilon}^{(3)}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap (\mathbb{S}_n \times \mathbb{T})} \mathbb{E}[\deg(P; \,\cdot\,) \mid P] \mathbb{1}\{U \leqslant 1/(\varepsilon n)\}$$

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \uparrow \infty} \mathbb{P}\Big(d_{\mathrm{Skorokhod}}(\overline{S}_{n,\varepsilon}^{(3)}, \overline{S}_n^{(2)}) > \delta\Big) = 0 \qquad \forall \delta > 0.$$

Step 4: Convergence of the low-mark edge count. [2]

$$\overline{S}_{n,\varepsilon}^{(3)} \xrightarrow[n \uparrow \infty]{d} S_\varepsilon^* \quad \text{in} \quad D([0,1], \mathbb{R})$$

Step 5: The limit $\lim_{\varepsilon \downarrow 0} S_\varepsilon^*(\,\cdot\,)$ exists in $D([0,1], \mathbb{R})$. [2] □

4. References

[1] P Gracar, A Grauer, L Lüchtrath, and P Mörters. The age-dependent random connection model. *Queueing Syst.*, 93:309-331, 2019.
[2] S. I. Resnick. *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*. Springer, 2007.

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